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Variational principle in dynamics of a vortex filament

Victor L. Berdichevsky
Mechanical Engineering, Wayne State University, Detroit, Michigan 48202, USA
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A variational principle governing the dynamics of a vortex filament in unbounded incompressible inviscid fluid flow was suggested by Rasetti and Regge [Physica A **80**, 217 (1975)]. This variational principle holds in the approximation taking into account the logarithmically large terms, on the order of $\ln(L/a)$, L and a being the length and the cross-section radius of the filament, respectively. In this approximation, the Hamiltonian is a function of L. Accordingly, the filament length L is constant in the course of motion. In this paper, a variational principle is obtained that takes into account also the terms on the order of unity. A characteristic feature of the more precise theory is the evolution of the filament length. The variational principle of the vortex filament dynamics is derived from the variational principle for the arbitrary vortex motion of an incompressible inviscid fluid found recently.

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I. INTRODUCTION

A vortex filament is a thin tube such that vorticity is negligible in the vicinity of this tube while inside the tube it is predominantly directed along the tube axis. In an inviscid fluid, the tube consists of the same particles in the course of motion. Motion of the vortex filament can be modeled by motion of a curve Γ. The dynamical equations for Γ are some equations for the position vector of Γ, $r^i(t,\eta)$, η being a parameter along Γ (latin indices run through the values 1,2,3 and correspond to projections on a Cartesian frame, x^i). Development of the dynamical equations describing the motion of Γ is an asymptotic problem; it employs the presence of a small geometric parameter α, the ratio of the characteristic size of the tube cross section to the characteristic curvature radius of the tube. This problem has been studied in detail (see [1], Chap. 11, for a historic review), and the dynamical equations governing the evolution of Γ are well established. We focus here on the case when the vortex filament has a circular cross section and moves in an inviscid incompressible unbounded fluid that is at rest at infinity; vorticity in the filament cross sections is assumed to be uniform.

An ideal fluid is a Hamiltonian system. Therefore, any special motion of the ideal fluid, in particular, the motion of a vortex filament, must be governed by a variational principle. The question under consideration is as follows: What is the variational principle that controls the dynamics of Γ? This question is not of pure academic interest. For example, to develop a numerical code which respects the symplectic structure of the equations and conserves energy, one has to know this variational principle.

Perhaps the first attempt to derive the equations of the vortex filament dynamics from a variational principle was undertaken by Rasetti and Regge [2]. They suggested the following action functional:

$$I = \int_{t_0}^{t_1} (\mathcal{A} - \mathcal{K})dt, \tag{1}$$

where $\mathcal{A}$ is a "shortened action," an analogy to the term $p\dot{q}$ of classical mechanics,

$$\mathcal{A} = \frac{\gamma}{3}\oint_\Gamma e_{ijk} r^j(t,\eta)\frac{\partial r^i(t,\eta)}{\partial t}\frac{\partial r^k(t,\eta)}{\partial \eta}d\eta, \tag{2}$$

and $\mathcal{K}$ the kinetic energy of the fluid,

$$\mathcal{K} = \frac{\gamma^2}{8\pi}\oint_\Gamma\oint_\Gamma \frac{1}{|\Delta r|}\frac{\partial r^i(t,\eta)}{\partial \eta}\frac{\partial r_i(t,\eta')}{\partial \eta'}d\eta\, d\eta'. \tag{3}$$

Here γ is the velocity circulation around the filament, e_{ijk} the Levi-Civita symbols, Δr the distance between the points η and η', $|\Delta r| \equiv \sqrt{\Delta r^i \Delta r_i}$, $\Delta r^i \equiv r^i(t,\eta) - r^i(t,\eta')$, and summation over repeated lower and upper indices is always implied.

The formula for the shortened action (2) was a remarkable finding: such a functional does not appear in other variational principles of fluid mechanics. However, the Rasetti-Regge variational principle has an ambiguity: the integral (3) diverges. The singularity in (3) is essential; it is quite different from that in point vortex theory. For a point vortex, the "infinite part" of the kinetic energy, the vortex self-energy, does not depend on the vortex position. Therefore, by dropping the infinite terms and keeping only the finite part of the kinetic energy, one obtains a sensible Hamiltonian dynamics. For vortex filaments, the singular part of the kinetic energy does depend on the filament positions. Moreover, it provides the leading contribution to the dynamical equations. The Hamiltonian structure of the dynamical equations in the case when one keeps only the leading logarithmically large terms[1] was established by Marsden and Weinstein [3] (see the review in [4], Sec. 6.3.B). This is the so-called self-induction approximation; it corresponds to a Hamiltonian that is proportional to the length of the filament, L,

$$\mathcal{K} = cL, \tag{4}$$

with L being a functional of the positions of Γ,

[1]That is, the terms on the order of $\ln \alpha$; in what follows we assume that the characteristic radius $\mathcal{R}$ of Γ is on the order of the length of the filament, L, and set α to be equal to the ratio of the cross-section radius a to L: $\alpha = a/L$.

$$L \equiv \oint_{\Gamma} \sqrt{r^{i}_{,\eta} r_{i,\eta}} d\eta. \tag{5}$$

The constant c is logarithmically large: it is proportional to $\ln \alpha$. The filament length L, being the Hamiltonian of the system, does not change in the course of motion. This is a characteristic feature of the self-induction approximation. The approximation is consistent in the sense that α and c become some constants in this approximation. The filament length changes if one takes into account the terms on the order of unity, and this is one of the motivations to consider a more precise theory.

In their paper [2], Rasetti and Regge suggested as plausible the following regularization of the diverging integral (3):

$$\mathcal{K} = \lim_{\lambda \to 0} \frac{\gamma^2}{8\pi} \left(\oint_{\Gamma} \oint_{\Gamma} \frac{1}{\sqrt{|\Delta r|^2 + \lambda^2}} \frac{\partial r^i(t,\eta)}{\partial \eta} \frac{\partial r_i(t,\eta')}{\partial \eta'} d\eta \, d\eta' + 2L \ln \frac{\lambda}{\lambda_0} \right), \tag{6}$$

where λ_0 is a parameter of regularization. Apparently, λ_0 was considered as a constant. The Hamiltonian (3) contains a term that is proportional to the filament length L and some additional terms. The constant λ_0 can be chosen to capture correctly the leading approximation (4). However, as further analysis shows, the additional terms in (6) do not provide a proper account for the terms of the next order, unity, if λ_0 is a constant.

The question arises: Could the functional $\mathcal{K}$ be chosen in such a way that the corresponding dynamical equations contain all the terms on the order of unity? We cannot exclude the possibility that for such a choice the meaning of $\mathcal{K}$ as the kinetic energy of the fluid motion can be lost. The very possibility of the existence of such a functional is far from being obvious. A doubt appears if one attempts to derive the Rasetti-Regge variational principle from the variational principle for general vortex flows found by Berdichevsky [5,6] and Kuznetsov and Ruban [7] (some further useful modifications of this variational principle and applications to statistical mechanics of vortex lines are given in [8,9]; see also the forthcoming monograph [10]; an extension to dynamics of vortex lines in compressible fluid was suggested by Ruban [11]). It turns out that the action functional of three-dimensional motion contains terms on the order of unity, which are not present in Eqs. (2) and (3), which casts doubts on the validity of the Rasetti-Regge principle in the next approximation.

This paper aims to address the above-mentioned issues. We derive the variational principle in the dynamics of vortex filaments from the variational principle of vortex line dynamics. To this end we approximate the functions describing the motion of vortex lines inside the vortex filament. The approximation involves, in addition to the positions of $\Gamma, r^i(t,\eta)$, two additional degrees of freedom: the cross-section radius a and the angle of rotation of cross sections, φ. In a sense, this method is similar to the derivation of the one-dimensional theory of elastic beams from three-dimensional elasticity theory [10]. The action functional is computed as a functional of $r^i(t,\eta)$, $a(t)$, and $\varphi(t)$ in the approximation accounting for the logarithmic terms and the terms on the order of unity. The action functional has the form (1) with $\mathcal{A}$ and $\mathcal{K}$ given by the formulas

$$\mathcal{A}(r,a,\varphi) = \frac{\gamma}{3} \oint_{\Gamma} e_{ijk} r^j(t,\eta) \frac{\partial r^i(t,\eta)}{\partial t} \frac{\partial r^k(t,\eta)}{\partial \eta} d\eta - \gamma \frac{a^2}{4} \frac{d\varphi}{dt} L, \tag{7}$$

$$\mathcal{K}(r,a) = \frac{\gamma^2}{8\pi} \oint_{\Gamma} \oint_{\Gamma} \frac{1}{|\Delta r| + \varepsilon} \frac{\partial r^i(t,\eta)}{\partial \eta} \frac{\partial r_i(t,\eta')}{\partial \eta'} d\eta \, d\eta' + \frac{\gamma^2}{8\pi} \left(2 \ln \frac{2\varepsilon}{a} + \frac{1}{2} \right) L. \tag{8}$$

Here ε is an arbitrary fixed parameter which is much smaller than L. Similarly to (6), one can consider in (8) the limit $\varepsilon \to 0$, but for practical applications this is not necessary. It is seen from (7) and (8) that the additional degrees of freedom bring terms on the order of unity into the action functional. The possibility of varying a independently of $r^i(t,\eta)$ is related to the compressibility of virtual motions of vortex lines. Thus, the volume of the filament, $\pi a^2 L$, for a virtual motion differs from that for the real motion.

It is shown further that the Hamiltonian equations corresponding to (7) and (8) coincide with the equations derived by the asymptotic analysis of the equations of ideal fluid. Moreover, $\mathcal{K}$ has the meaning of the kinetic energy of fluid motion.

Remarkably, the variational principle obtained admits the elimination of the two degrees of freedom a and φ. Indeed, the kinetic energy does not depend on φ, and the variation of φ yields conservation of the volume of the filament:

$$\frac{d}{dt} a^2 L = 0, \tag{9}$$

or

$$\pi a^2 L = \mathring{V}, \tag{10}$$

$\mathring{V}$ being the volume of the filament. The variation of a results in the equation determining the cross-section rotation rate

$$\frac{d\varphi}{dt} = \frac{\gamma}{2\pi a^2}. \tag{11}$$

Equation (11) shows that the cross-section rate coincides with vorticity inside the filament. Equation (10) allows one to express a in terms of L, i.e., as a functional of the filament positions . After elimination of a and φ, we arrive at the

Rasetti-Regge variational principle in which $\mathcal{K}$ is obtained from (8) by substitution of a from (10), $a=\sqrt{\mathring{V}/\pi L}$:

$$\mathcal{K}(r) = \frac{\gamma^2}{8\pi}\oint_\Gamma \oint_\Gamma \frac{1}{|\Delta r|+\varepsilon}\frac{\partial r^i(t,\eta)}{\partial \eta}\frac{\partial r_i(t,\eta')}{\partial \eta'}d\eta\, d\eta' + \frac{\gamma^2}{8\pi}\left(\ln\frac{4\pi\varepsilon^2 L}{\mathring{V}}+\frac{1}{2}\right)L. \tag{12}$$

We see that $\mathcal{K}$ still has the structure of the functional (6), if one admits that λ_0 in (6) is not a constant but a function of L. We emphasize that this is a feature of the approximation considered, and, most probably, in the next approximations it will be lost along with the possibility of excluding the terms with internal degrees of freedom. In principle, the result formulated can be derived by computing the kinetic energy that incorporates all the terms on the order of unity and showing by direct inspection that it yields the known dynamical equations. Then, however, one has to explain why the terms of the same order in the action functional, like, e.g., the last term in (7), can be ignored.

In the next section we formulate the dynamical equations that follow from the asymptotic analysis of an the equations of an ideal fluid. Then in Sec. III we formulate the variational principle for vortex flows. The filament kinematics is introduced in Sec. IV. In Secs. V and VI we find the leading asymptotic terms of the kinetic energy and functional $\mathcal{A}$. In Sec. VII we formulate the variational principle and derive the equations of vortex filament dynamics and show their asymptotic equivalence to the equations of Sec. II.

II. DYNAMICAL EQUATIONS OF VORTEX FILAMENT

To formulate a closed system of equations for $r^i(t,\eta)$, we note the formula for the velocity $u^i(s,t)$ of the points of Γ (see [1]):

$$u^i(s,t) = \frac{\gamma}{4\pi}e^{ijk}\int_{\rho(s',s)\geq a\delta}\frac{\tau_j(s',t)[r_k(s,t)-r_k(s',t)]}{|\Delta r|^3}ds'. \tag{13}$$

Here s is the arclength of Γ, τ^i the components of the tangent vector, and $\rho(s',s)$ the shortest distance between the points s' and s measured along Γ. The integral (13) contains a cutoff of the radius $a\delta,\delta$ being a numerical factor on the order of unity. For filaments with uniform vorticity $\ln 2\delta = 1/4$. In Eq. (13) the terms of order α are neglected compared to unity and the logarithmic terms are treated as terms of order unity; the same accuracy will be maintained in all further relations.

It was argued by Moore and Saffman [12] (see also [1], Sec. 11.4) that the cross-section radius a is constant along the filament to leading order, i.e., $a=a(t)$.

Formula (13) can be rewritten in terms of an arbitrary parameter η along Γ, since η and s are linked by the relation

$$ds = \sqrt{r^i_{,\eta} r_{i,\eta}}\,d\eta, \quad r^i_{,\eta} \equiv \frac{\partial r^i(t,\eta)}{\partial \eta}.$$

Formula (13) shows that the velocity u^i at a point η is determined uniquely by the current position of the filament, $r^i(\eta)$, and the current cross-section radius a; we write this fact symbolically as $u^i = u^i(\eta|\mathbf{r}(\eta),a)$. The tangent velocity $\tau_i u^i$ causes motion of Γ over itself. Thus, it is irrelevant to the dynamics of Γ, and only the normal velocity yields change of the position of Γ. Equating the normal components of $\partial r^i/\partial t$ and u^i, we obtain the equations governing the dynamics of a vortex filament:

$$e_{ijk}\left(\frac{\partial r^j(t,\eta)}{\partial t} - u^i(\eta|\mathbf{r}(t,\eta),a)\right)\tau^k = 0. \tag{14}$$

To make Eqs. (14) closed one has to link the cross-section radius a with the required functions $r^j(t,\eta)$. Since the vortex filament consists of the same fluid particles, and the fluid is incompressible, the volume of the vortex filament does not change in the course of motion. Therefore, a can be determined by the incompressibility condition (10). Equations (14), (10), and (5) form a closed system of equations for dynamics of the vortex filament.

III. VARIATIONAL PRINCIPLE OF VORTEX LINE DYNAMICS

Consider motion of an ideal incompressible fluid in a closed vessel V. For simplicity, let all vortex lines be closed. We introduce Lagrangian coordinates in such a way that X^3 coordinate lines coincide with vortex lines while each vortex line is marked by two Lagrangian coordinates $(X^1,X^2)\equiv X$. Denote by $r^i(t,\eta,X)$ the position vector of a vortex line with Lagrangian coordinates X and a parameter along the vortex line, η. Let $\mathring{\omega}(X)$ be the only nonzero component of vorticity in Lagrangian coordinates. The kinetic energy $\mathcal{K}$ can be computed as a functional of vortex line positions $r^i(t,\eta,X)$: $\mathcal{K} = K(r^i(t,\eta,X))$ (the reader is referred to [5,10] for more details). The following variational principle holds: the true motion of vortex lines is a stationary point of the action functional (1) where $\mathcal{K}$ is the kinetic energy of the fluid while the functional $\mathcal{A}$ is

$$\mathcal{A} = \frac{1}{3}\int_V e_{ijk} r^j(t,\eta,X)\frac{\partial r^i(t,\eta,X)}{\partial t}\frac{\partial r^k(t,\eta,X)}{\partial \eta}\mathring{\omega}(X)d^2X\, d\eta. \tag{15}$$

An important peculiarity of the variational principle is that fluid motion inside the vortex region is allowed to be compressible in full agreement with the independence of the vortex dynamics on the motion of particles over the vortex lines.

IV. FILAMENT KINEMATICS

Let us attach to each point of Γ a pair of vectors $\tau_1^i(t,\eta)$ and $\tau_2^i(t,\eta)$, which form together with τ^i an orthonormal triad. This introduces one additional degree of freedom, the angle of rotation of the triad around τ^i. The rate of rotation is characterized by a scalar[2]

$$\Omega(t,\eta) = \frac{1}{2} e^{\mu\nu} \tau_{i\nu} \frac{\partial \tau^j_\mu}{\partial t}. \quad (16)$$

The triad vectors obey the relations

$$\tau_i \frac{\partial \tau^j_\mu}{\partial t} + \tau^j_\mu \frac{\partial \tau_i}{\partial t} = 0, \quad \tau_{i\nu} \frac{\partial \tau^j_\mu}{\partial t} + \tau_{i\mu} \frac{\partial \tau^j_\nu}{\partial t} = 0. \quad (17)$$

They follow from differentiation of the orthogonality conditions $\tau_i \tau^j_\mu = 0$, $\tau^j_\mu \tau_{i\nu} = \delta_{\mu\nu}$. Projecting the derivatives $\partial \tau^j_\mu / \partial t$ on the triad vectors and using the relations (17), we obtain a system of differential equations for the two triad vectors τ^j_μ,

$$\frac{\partial \tau^j_\mu(t,\eta)}{\partial t} = -\left(\frac{\partial \tau^j(t,\eta)}{\partial t} \tau_{j\mu} \right) \tau^j + \Omega(t,\eta) e^{\cdot\nu}_{\mu\cdot} \tau^j_\nu. \quad (18)$$

In this system, $\partial \tau^j / \partial t$ and τ^j are known, if one knows the motion of Γ. Therefore, the evolution of the two triad vectors $\tau^j_\mu(t,\eta)$ is uniquely determined, if the motion of Γ, the rotation rate Ω, and the initial conditions are given.

We make a special choice of τ^j_μ. First, we choose some initial vectors $\tau^j_\mu(0,\eta)$ and specify the evolution of τ^j_μ by setting $\Omega = 0$. We denote these uniquely defined vectors by $\overset{\circ}{\tau}{}^j_\mu(t,\eta)$. Then we introduce the vectors $\tau^j_\mu(t,\eta)$ which differ from $\overset{\circ}{\tau}{}^j_\mu$ by a rotation on the same angle at each point of Γ:

$$\tau^j_\mu(t,\eta) = o^\nu_\mu(t) \overset{\circ}{\tau}{}^j_\nu(t,\eta). \quad (19)$$

Here $o^\nu_\mu(t)$ is a two-dimensional orthogonal matrix. Every two-dimensional orthogonal matrix is in one-to-one correspondence with the angle of rotation. Denoting by $\varphi(t)$ the angle of rotation corresponding to the matrix $o^\nu_\mu(t)$, we find from (16) and (19) that

$$\Omega = \frac{d\varphi(t)}{dt}.$$

Hence, the rotation rate Ω is constant over Γ. We emphasize that Ω may change in time. As we will see further, the parameter Ω coincides with vorticity inside the vortex tube at the stationary points of the action functional.

So, in addition to $r^i(t,\eta)$, the vortex filament is endowed with two degrees of freedom, $a(t)$ and the angle of rotation of the filament cross sections, $\varphi(t)$.

The vectors τ^j_μ are linked to dynamics of the fluid by the following ansatz: the positions of vortex lines inside the vortex filament can be approximated by linear functions of the cross-sectional Lagrangian coordinates X^μ:

$$r^i(t,\eta,X) = r^i(t,\eta) + a(t) \tau^j_\mu(t,\eta) X^\mu. \quad (20)$$

The Lagrangian coordinates X^μ are assumed to be changed within a unit circle: $X^\mu X_\mu \leq 1$. This ansatz is far from being obvious. Its adequacy is justified by the fact that, as we will see, it yields Eqs. (14) and (10), which are firmly established by the asymptotic analysis of the equations of an ideal incompressible fluid.

V. KINETIC ENERGY

Let the flow be unbounded and the fluid be at rest at infinity. If the vorticity ω^i is not equal to zero only in some bounded region V, then the kinetic energy can be found explicitly (see, e.g., [13], Sec. 7.2):

$$\mathcal{K} = \frac{1}{2} \int_V \int_V \frac{\omega^i(x)\omega_i(x')}{4\pi |x - x'|} d^3x\, d^3x'. \quad (21)$$

Here x is a point of V, $|x-x'|$ the distance between the points x and x', and d^3x the volume element of V. We suppress in this section the dependence on time. We have to compute kinetic energy of the motions (20). For such motions, the region V may be thought of as the region swept by a circle of radius a moving over Γ. The circle is centered at Γ. The vorticity in the leading approximation is $\omega^i = \gamma \tau^i / \pi a^2$.

It is convenient to introduce an auxiliary parameter of the dimension of length, ε, which is much larger than a and much smaller than the characteristic radius of Γ. We are going to show that, for any such ε, the leading terms of the kinetic energy are given by formula (8). It will be seen from the derivation that, in fact, $\mathcal{K}$ does not depend on the choice of ε: variations of the two terms of (8) caused by the change of ε cancel out in the leading approximation and affect only the terms neglected in (8).

To prove (8) we split the integral (21) into a sum of two integrals:

$$\mathcal{K} = \frac{1}{2} \int_V \int_V \frac{\omega^i(x)[\omega_i(x') - \omega_i(x)]}{4\pi |x - x'|} d^3x\, d^3x' + \frac{\gamma^2}{2(\pi a^2)^2} \int_V \int_V \frac{d^3x\, d^3x'}{4\pi |x - x'|}. \quad (22)$$

The first integral is not singular. As $a \to 0$, it converges to

$$\frac{\gamma^2}{2} \oint_\Gamma \oint_\Gamma \frac{\tau^i(s)[\tau_i(s') - \tau_i(s)]}{4\pi |\Delta r|} ds\, ds'.$$

The second integral is again split into a sum of two integrals: for some ℓ such that $a \ll \ell \ll \mathcal{R}$, we present the second integral in the form

$$\frac{\gamma^2}{2(\pi a^2)^2} \int_V \int_{V,\rho(s',s)\geq \ell} \frac{d^3x\, d^3x'}{4\pi |x - x'|} + \frac{\gamma^2}{2(\pi a^2)^2} \int_V \int_{V,\rho(s',s)\leq \ell} \frac{d^3x\, d^3x'}{4\pi |x - x'|}. \quad (23)$$

The first integral in (23) is not singular as $a \to 0$ and converges to

$$\frac{\gamma^2}{2} \oint_\Gamma \oint_{\rho(s',s)\geq \ell} \frac{ds\, ds'}{4\pi |\Delta r|}.$$

The second integral in (23) is

[2] $e^{\mu\nu}$ is the two-dimensional Levi-Cività symbol, greek indices μ, ν, λ run through values 1,2 and mark projections on τ^j_μ; the local basis $\{\tau^j, \tau^j_1, \tau^j_2\}$ is Cartesian, therefore the tensor components with upper and lower indices i,j,k and μ,ν coincide.

$$\frac{\gamma^2}{8\pi^3}\oint_\Gamma ds\int_{X^\mu X_\mu\leq 1}\int_{X'^\mu X'_\mu\leq 1} 2\int_0^\ell \frac{d\xi\, d^2X\, d^2X'}{\sqrt{a^2(X^\mu - X'^\mu)(X_\mu - X'_\mu)+\xi^2}}.$$

Integrating over ξ and using that $\ell \gg a$, we have for this integral

$$\frac{\gamma^2}{8\pi^3}\oint_\Gamma ds\int_{X^\mu X_\mu\leq 1}\int_{X'^\mu X'_\mu\leq 1} 2\ln\frac{2\ell}{a|X-X'|}d^2X\, d^2X'$$
$$=\frac{\gamma^2}{8\pi^3}\oint_\Gamma ds\, 2\left(\pi^2\ln\frac{2\ell}{a}+J\right),$$

$$J \equiv \int_{X^\mu X_\mu\leq 1}\int_{X'^\mu X'_\mu\leq 1}\ln\frac{1}{|X-X'|}d^2X\, d^2X'.$$

The number J as is easy to see, is equal to $\pi^2/4$. Collecting the results, we obtain

$$\mathcal{K}=\frac{\gamma^2}{2}\oint_\Gamma\oint_\Gamma\frac{\tau^i(s)[\tau_i(s')-\tau_i(s)]}{4\pi|\Delta r|}ds\, ds'$$
$$+\frac{\gamma^2}{2}\oint_\Gamma\oint_{\rho(s',s)\geq\ell}\frac{ds\, ds'}{4\pi|\Delta r|}+\frac{\gamma^2}{8\pi}\left(2\ln\frac{2\ell}{a}+\frac{1}{2}\right)L.$$

This formula can be simplified further. Indeed, in the first term the integral over s' may be replaced by the integral over such s' that $\rho(s',s)\geq\ell$, because the integral over the region, $\rho(s',s)\leq\ell$, is negligible within the accepted accuracy. Thus,

$$\mathcal{K}=\frac{\gamma^2}{8\pi}\oint_\Gamma\oint_{\rho(s',s)\geq\ell}\frac{\tau^i(s)\tau_i(s')}{|\Delta r|}ds\, ds'+\frac{\gamma^2}{8\pi}\left(2\ln\frac{2\ell}{a}+\frac{1}{2}\right)L. \tag{24}$$

On the other hand, for $\varepsilon \ll \ell$,

$$\frac{\gamma^2}{8\pi}\oint_\Gamma\oint_\Gamma\frac{\tau^i(s)\tau_i(s')}{|\Delta r|+\varepsilon}ds\, ds'$$
$$=\frac{\gamma^2}{8\pi}\oint_\Gamma\int_{\rho(s',s)\geq\ell}\frac{\tau^i(s)\tau_i(s')}{|\Delta r|+\varepsilon}ds\, ds'$$
$$+\frac{\gamma^2}{8\pi}\oint_\Gamma\int_{\rho(s',s)\leq\ell}\frac{\tau^i(s)\tau_i(s')}{|\Delta r|+\varepsilon}ds\, ds'.$$

In the first integral, ε can be dropped. In the second integral, expanding $\tau_i(s')$ in Taylor series in vicinity of the point s, we see that only the first term of the expansion provides a noticeable contribution. Therefore,

$$\frac{\gamma^2}{8\pi}\oint_\Gamma\oint_\Gamma\frac{\tau^i(s)\tau_i(s')}{|\Delta r|+\varepsilon}ds\, ds'=\frac{\gamma^2}{8\pi}\oint_\Gamma\int_{\rho(s',s)\geq\ell}\frac{\tau^i(s)\tau_i(s')}{|\Delta r|}$$
$$\times ds\, ds'+\frac{\gamma^2}{8\pi}L2\ln\frac{\ell}{\varepsilon}. \tag{25}$$

Formula (8) follows from (24) and (25).

VI. FUNCTIONAL $\mathcal{A}$

The direct computation of the functional $\mathcal{A}$ (15) for the motions (20) is a cumbersome task. A simpler way is to employ the relation which holds for any variation of the functional $\mathcal{A}$ (15):

$$\delta\int_{t_0}^{t_1}\mathcal{A}\, dt=\int_{t_0}^{t_1}\int_V e_{ijk}\frac{\partial r^i(t,\eta,X)}{\partial t}\frac{\partial r^k(t,\eta,X)}{\partial\eta}$$
$$\times\delta r^j(t,\eta,X)\mathring{\omega}(X)d^2X\, d\eta\, dt.$$

Plugging in (20) and using that $\mathring{\omega}=\gamma/\pi$, we have

$$\delta\int_{t_0}^{t_1}\mathcal{A}\, dt=\gamma\int_{t_0}^{t_1}\left[\oint_\Gamma\left(e_{ijk}\frac{\partial r^i}{\partial t}\tau^k+\frac{d\varphi}{dt}\frac{a^2}{4}\frac{d\tau_j}{ds}\right)\delta r^j ds\right.$$
$$\left.+\frac{1}{4}\frac{d(a^2L)}{dt}\delta\varphi-\frac{aL}{2}\frac{d\varphi}{dt}\delta a\right]dt. \tag{26}$$

It follows from (26) that, up to divergence terms, the functional $\mathcal{A}$ is given by formula (7).

VII. VARIATIONAL PRINCIPLE AND DYNAMICAL EQUATIONS

We claim that the following variational principle holds: the true motion of a vortex filament is a stationary point of the action functional (1) on the set of functions $r^i(t,\eta)$, $a(t)$, and $\varphi(t)$, if $\mathcal{A}$ and $\mathcal{K}$ are defined by formulas (7) and (8), respectively. To derive the corresponding dynamical equations, we find the variation of the kinetic energy (8),

$$\delta\int_{t_0}^{t_1}\mathcal{K}\, dt=\int_{t_0}^{t_1}\left(\oint_\Gamma\frac{\delta\mathcal{K}}{\delta r^j}\delta r^j ds-\frac{\gamma^2 L}{4\pi a}\delta a\right)dt,$$

$$\frac{\delta\mathcal{K}}{\delta r^j}=\gamma e_{ijk}v^i\tau^k-\frac{\gamma^2}{8\pi}\frac{d\tau_j}{ds}\left(2\ln\frac{2\varepsilon}{a}+\frac{1}{2}\right). \tag{27}$$

Here the following notation was used:

$$v^i\equiv\frac{\gamma}{4\pi}e^{ijk}\oint_\Gamma\frac{\tau_j(t,s')\Delta r_k}{(|\Delta r|+\varepsilon)^2|\Delta r|}ds'. \tag{28}$$

The derivative $d\tau_j/ds$ can also be written as $-e_{ijk}b^i\tau^k/R$, where b^i is the binormal and R the curvature radius. Therefore,

$$\frac{\delta\mathcal{K}}{\delta r^j}=\gamma e_{ijk}\left[v^i+\frac{\gamma}{8\pi}\frac{b^i}{R}\left(2\ln\frac{2\varepsilon}{a}+\frac{1}{2}\right)\right]\tau^k.$$

Equating (27) and (26), we obtain the governing equations of the vortex filament dynamics:

$$e_{ijk}\left[\frac{\partial r^i(t,\eta)}{\partial t}-v^i-\frac{\gamma}{8\pi}\frac{b^i}{R}\left(2\ln\frac{2\varepsilon}{a}+\frac{1}{2}\right)-\frac{a^2}{4}\frac{d\varphi}{dt}\frac{b^i}{R}\right]\tau^k=0, \tag{29}$$

$$\frac{d\varphi}{dt}=\frac{\gamma}{2\pi a^2},\qquad \frac{d}{dt}a^2L=0. \tag{30}$$

The first equation (30) determines the angular velocity of the cross sections, the second one means conservation of the filament volume.

Let us show that Eqs. (14), (10), (29), and (30) are asymptotically equivalent. To this end, we first justify the relation

$$v^i + \frac{\gamma^2}{8\pi}\frac{b^i}{R}\left(2\ln\frac{2\varepsilon}{a} + \frac{1}{2}\right) = \frac{\gamma}{4\pi}e^{ijk}\int_{\rho(s',s)\geq\ell}\frac{\tau_j(t,s')\Delta r_k}{|\Delta r|^3}ds' + \frac{\gamma^2}{8\pi}\frac{b^i}{R}\left(2\ln\frac{2\ell}{a} - \frac{3}{2}\right). \quad (31)$$

This relation can be obtained by splitting the integral in (28) into a sum of integrals over regions $\rho(s',s)\geq\ell$ and $\rho(s',s)\leq\ell$. In the integral over $\rho(s',s)\geq\ell$ the parameter ε can be dropped. The integral over $\rho(s',s)\leq\ell$ is equal to $\frac{\gamma}{4\pi}\frac{b^i}{R}(\ln\frac{\ell}{\varepsilon}-1)$ as follows from the equalities

$$\frac{\gamma}{4\pi}e^{ijk}\int_{\rho(s',s)\leq\ell}\frac{\tau_j(t,s')\Delta r_k}{(|\Delta r|+\varepsilon)^2|\Delta r|}ds'$$
$$= \frac{-\gamma}{4\pi}2\int_0^\ell \frac{e^{ijk}\left(\tau_j + \frac{d\tau_j}{ds}\xi\right)\left(\tau_k\xi + \frac{1}{2}\frac{d\tau_k}{ds}\xi^2\right)}{(\xi+\varepsilon)^2\xi}d\xi$$
$$= \frac{-\gamma}{4\pi}\int_0^\ell \frac{e^{ijk}(d\tau_j/ds)\tau_k\xi}{(\xi+\varepsilon)^2}d\xi = \frac{\gamma}{4\pi}\frac{b^i}{R}\int_0^\ell\frac{\xi}{(\xi+\varepsilon)^2}d\xi$$
$$= \frac{\gamma}{4\pi}\frac{b^i}{R}\left(\ln\frac{\ell}{\varepsilon} - 1\right).$$

Plugging these results into the left-hand side of (31), we see that (31) holds true.

Let us put now (13) into a form convenient for comparison with (29). We split the integral in (13) into the sum of integrals over $\rho(s',s)\geq\ell$ and $a\delta\leq\rho(s',s)\leq\ell$. Using that $a\ll\ell\ll\mathcal{R}$, we calculate the second integral by expanding $\tau_j(s',t)$ and $r_k(s',t)$ in Taylor's series in the vicinity of the point s:

$$\frac{\gamma}{4\pi}\left(\int_{-\ell}^{-a\delta} + \int_{a\delta}^{\ell}\right)\frac{-e^{ijk}\left(\tau_j + \frac{d\tau_j}{ds}\xi\right)\left(\tau_k\xi + \frac{1}{2}\frac{d\tau_k}{ds}\xi^2\right)}{|\xi|^3}d\xi$$
$$= \frac{\gamma}{4\pi}e^{ijk}\tau_j\frac{d\tau_k}{ds}\int_{a\delta}^{\ell}\frac{d\xi}{\xi} = \frac{\gamma b^i}{4\pi R}\ln\frac{\ell}{a\delta}.$$

Therefore,

$$u^i(s,t) = \frac{\gamma}{4\pi}e^{ijk}\int_{\rho(s',s)\geq\ell}\frac{\tau_j(s',t)(r_k(s,t) - r_k(s',t))}{|\Delta r|^3}ds' + \frac{\gamma}{4\pi}\frac{b^i}{R}\ln\frac{\ell}{a\delta}.$$

Recalling that $\ln 2\delta = 1/4$, we see that Eqs. (14) and (29) coincide indeed. That confirms the validity of the variational principle formulated.

Formulas (7) and (8) show that, after the change of variables, $a\to V$, $V=\pi a^2 L$, the couple (V,φ) has the meaning of action-angle variables (see [14]). Due to the special form of the functional $\mathcal{A}$ (7), these variables can be eliminated, and the variational principle holds: the true trajectories of the vortex filament are the stationary points of the action functional (1) on the set of functions $r^i(t,\eta)$, if $\mathcal{A}$ is defined by formula (2) while $\mathcal{K}$ is the functional (12).

For the kinetic energy one can use a Rosenhead-type approximation, which does not employ an auxiliary parameter ε,

$$\mathcal{K} = \frac{\gamma^2}{8\pi}\oint_\Gamma\oint_\Gamma\frac{\tau^i(s)\tau_i(s')}{|\Delta r| + \mu\sqrt{\mathring{V}/L}}ds\,ds'. \quad (32)$$

It is easy to see that (12) and (32) are asymptotically equivalent, if the factor μ is defined by the relation $\ln(2\mu\sqrt{\pi}) = -1/4$. The Hamiltonian system with energy (32) is quite complex, because it involves the parameter L, which varies, though asymptotically this system is equivalent to the considered one. This is why a constant parameter ε has been used.

If we neglect in (12) the terms on the order of unity, we obtain for the kinetic energy the expression

$$\mathcal{K} = \frac{\gamma^2}{8\pi}L\ln\frac{4\pi\varepsilon^2 L}{\mathring{V}}$$

or

$$\mathcal{K} = \frac{\gamma^2}{8\pi}L\ln L + \text{const},$$

which is asymptotically equivalent to (4).

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